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The adjoint of a positive semigroup

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0. Introduction

Let $\{T(t)\}_{t\geqslant 0}$ be a C_0-semigroup of bounded linear operators on a (real or complex) Banach space X. By defining $T^*(t) := (T(t))^*$ for each t, one obtains a semigroup $\{T^*(t)\}_{t\geqslant 0}$ on the dual space X^*. Throughout this paper, we will denote the semigroups $\{T(t)\}_{t\geqslant 0}$ and $\{T^*(t)\}_{t\geqslant 0}$ by $T(t)$ and $T^*(t)$, respectively, and it will be clear from the context when we mean the semigroup or the single operator.

The adjoint semigroup $T^*(t)$ fails in general to be strongly continuous again. Therefore it makes sense to define

$$X^{\odot} = \left\{x^* \in X^* : \lim_{t\downarrow 0} \|T^*(t)x^* - x^*\| = 0\right\}.$$

This is the maximal subspace of X^* on which $T^*(t)$ acts in a strongly continuous way. The space $X^{\odot}$ was introduced by Phillips in 1955 [Ph]. Recently, this space has been studied extensively by various authors (e.g., [Ne], [NP], [P]), in particular in connection with applications to certain evolution equations (e.g., [Cl]).

The purpose of this paper is to study the properties of $E^{\odot}$ in case E is a Banach lattice and $T(t)$ is a *positive* C_0-semigroup. Virtually nothing is known about the Banach lattice properties of $E^{\odot}$ and one of the most obvious questions, viz. under what conditions $E^{\odot}$ is a sublattice of E^*, is wide open. If $T^*(t)$ is a lattice semigroup, in particular if $T(t)$ extends to a positive group, then $E^{\odot}$ is a sublattice [Cl, Part IV]; this follows from

$$|T^*(t)(x^{\odot})^+ - (x^{\odot})^+| = |(T^*(t)x^{\odot})^+ - (x^{\odot})^+| \leqslant |T^*(t)x^{\odot} - x^{\odot}|$$

and the lattice property of the norm. Recently, Grabosch and Nagel [GN] constructed a positive C_0-semigroup on an AL-space E for which $E^{\odot}$ is not a sublattice of E^*. In fact, in this example the space $E^{\odot}$, with the order inherited from E^*, even fails to be a Banach lattice in its own right.

In order to motivate our main results, we start by considering in some detail the translation group $T(t)$ on $C_0(\mathbb{R})$, given by $T(t)f(s) = f(t+s)$. This semigroup has some features which turn out to be representative for the abstract situation.

THEOREM 0.1. *Let $T(t)$ be the translation group on $E = C_0(\mathbb{R})$.*

(i) ([P1]) *$\mu \in E^{\odot}$ if and only if μ is absolutely continuous with respect to the Lebesgue measure m.*

(ii) ([MG], [WY]) *If $\mu \in E^*$ is singular with respect to m, then $T(t)\mu \perp \mu$ for almost all $t \in \mathbb{R}$. In particular, for any $\nu \in E^*$ we have* $\limsup_{t\downarrow 0} \|T^*(t)\nu - \nu\| = 2\|\nu_s\|$, *where ν_s is the singular part of ν.*

(iii) *The space of singular measures is $T^*(t)$-invariant.*

Note that $T^*(t)\nu$ is just the translate in the opposite direction of ν in the sense that for a measurable set G we have $(T^*(t)\nu)(G) = \nu(G+t)$. Also, by (i) it is clear that a measure μ is singular if and only if $\mu \perp E^{\odot}$ in the Banach lattice sense. Versions of Theorem 0.1 for commutative locally compact groups (instead of $\mathbb{R}$) can be found in [GM, Chapter 8]. In [Pa2], the Wiener-Young theorem ((ii) above) has been analysed in detail in the context of adjoint semigroups. There, extensions have been obtained for the adjoints of positive semigroups essentially on $C(K)$-spaces. In the present paper, most of the results in [Pa2] will be extended to positive semigroups on arbitrary Banach lattices. For the convenience of the reader, we include full proofs. Although several proofs are completely different, this causes a small overlap with [Pa2].

We will prove the following Banach lattice versions of (i)-(iii). Let $T(t)$ be a positive C_0-semigroup on a Banach lattice E. Then:

(i) $E^{\odot}$ is a projection band if E^* has order continuous norm (Theorem 2.1).

The most important class of (non-reflexive) Banach lattices whose duals have order continuous norm is the class of AM-spaces. This class contains $C_0(\mathbb{R})$. In contrast, note that the dual of an AL-space does not have order continuous norm unless E is finite-dimensional.

(ii) Suppose $x^* \perp E^{\odot}$. Then we have $\limsup_{t\downarrow 0} \|T^*(t)x^* - x^*\| \geqslant 2\|x^*\|$ (Theorem 4.4). If moreover E^* has order continuous norm or E has a quasi-interior point, then $T^*(t)x^* \perp x^*$ for almost all $t \geqslant 0$ (Corollary 3.4).

(iii) The disjoint complement of $E^{\odot}$ is $T^*(t)$-invariant if $T^*(t)$ is a lattice semigroup (Corollary 4.8).

We use (iii) to show that if $T^*(t)$ is a lattice semigroup, then the quotient $E^*/(E^{\odot})^{dd}$ is either zero or else 'very large' (Theorem 4.10). Here $(E^{\odot})^{dd}$ is the band generated by $E^{\odot}$.

We assume the reader to be familiar with some standard theory of Banach lattices. For more information as well as the terminology we refer to [M], [AB], [S], [Z]. Throughout this paper, all Banach spaces and lattices may be either real or complex.

1. Some preliminary information

In this section we recall some of the basic facts about adjoint semigroups which will be used in the sequel. Proofs can be found e.g. in [BB].

Let $T(t)$ be a C_0-semigroup (i.e., a strongly continuous semigroup) on a Banach space X. Its generator will be denoted by A with domain $D(A)$. Considering the adjoint semigroup $T^*(t)$ on the dual space X^*, we define

$$X^{\odot} = \left\{x^* \in X^* \colon \lim_{t \downarrow 0} \|T^*(t)x^* - x^*\| = 0\right\},$$

the domain of strong continuity of $T^*(t)$. Then $X^{\odot}$ is a $T^*(t)$-invariant, norm closed, weak*-dense subspace of X^* (hence $X^{\odot} = X^*$ if X is reflexive). The space $X^{\odot}$ is precisely the norm closure of $D(A^*)$, the domain of the adjoint of A. In particular, for $\lambda \in \rho(A) = \rho(A^*)$ we have $R(\lambda, A^*)x^* \in X^{\odot}$ for all $x^* \in X^*$, where $R(\lambda, A^*) = R(\lambda, A)^* = (\lambda - A^*)^{-1}$ is the resolvent. For all $x^* \in X^*$ we have $\lim_{\lambda \to \infty} \lambda R(\lambda, A^*)x^* = x^*$, where the limit is in the weak*-sense. An alternative description of $X^{\odot}$ is given by

$$X^{\odot} = \left\{x^* \in X^* \colon \lim_{\lambda \to \infty} \|\lambda R(\lambda, A^*)x^* - x^*\| = 0\right\}.$$

If $T(t)$ extends to a C_0-group, then the space $X^{\odot}$ with respect to the semigroup $\{T(t)\}_{t \geq 0}$ is equal to the domain of strong continuity of the group $\{T(t)\}_{t \in \mathbb{R}}$.

Examples of spaces $X^{\odot}$ for various semigroups can be found in [BB], [Ne], [NP]. In particular we mention that if $T(t)$ is the translation group on $C_0(\mathbb{R})$ or $L^1(\mathbb{R})$, then $X^{\odot}$ can be identified canonically with $L^1(\mathbb{R})$ and $BUC(\mathbb{R})$ (the space of all bounded, uniformly continuous functions on $\mathbb{R}$), respectively.

We will have the occasion to use the so-called weak*-integrals (or Gelfand integrals) of X^*-valued functions. Let $[a, b] \subset \mathbb{R}$ and $f \colon [a, b] \to X^*$ a weak*-continuous function (or, more generally, a weak*-measurable function such that $t \mapsto \langle f(t), x\rangle \in L^1[a, b]$ for all $x \in X$). The weak*-integral weak* $\int_a^b f(t)\,dt \in X^*$ is then defined by the formula

$$\left\langle \text{weak*} \int_a^b f(t)\,dt, x \right\rangle = \int_a^b \langle f(t), x\rangle\,dt, \quad \forall x \in X.$$

In this situation, the function $t \mapsto \|f(t)\|$ is a Borel function on $[a, b]$ and we have the estimate

$$\left\| \text{weak*} \int_a^b f(t)\,dt \right\| \leqslant \int_a^b \|f(t)\|\,dt.$$

If $T(t)$ is a C_0-semigroup on X, then for each $x^* \in X^*$ the map $t \mapsto T^*(t)x^*$ is weak*-continuous on $[0, \infty)$ and for all $0 \leqslant a < b \in \mathbb{R}$ we have

$$\text{weak}^* \int_a^b T^*(t)x^* \, dt \in D(A^*) \subset X^\odot.$$

Finally we say a few words about the Banach lattice situation. Let E be a Banach lattice and $T(t)$ a positive C_0-semigroup on E. Suppose that M, ω are such that $\|T(t)\| \leqslant Me^{\omega t}$ for all $t \geqslant 0$. If $\lambda \in \mathbb{R}$ is such that $\lambda > \omega$, then $\lambda \in \rho(A)$ and $R(\lambda, A) \geqslant 0$ (for the basic theory of positive semigroups we refer to [Na]). As mentioned in the introduction, $E^\odot$ need not be a sublattice of E^*. As usual, we denote by $(E^\odot)^d$ the disjoint complement of $E^\odot$ in E^*, i.e.,

$$(E^\odot)^d = \{x^* \in E^* : x^* \perp y^\odot \text{ for all } y^\odot \in E^\odot\}.$$

Here $x^* \perp y^\odot$ means that $|x^*| \wedge |y^\odot| = 0$. Then $(E^\odot)^{dd}$, the disjoint complement of $(E^\odot)^d$, is equal to the band generated by $E^\odot$. Since $E^\odot = \overline{D(A^*)}$, it is clear that $(E^\odot)^{dd} = (D(A^*))^{dd}$. In general, $(E^\odot)^d$ is not $T^*(t)$-invariant (see Example 3.7). However, the subspace $(E^\odot)^{dd}$ is always $T^*(t)$-invariant. Indeed, if $x^* \in E^*$ is such that $|x^*| \leqslant |R(\lambda, A^*)y^*|$ for some $y^* \in E^*$ and $\lambda > \omega$, then $|T^*(t)x^*| \leqslant R(\lambda, A^*)T^*(t)|y^*|$. This shows that the (order) ideal generated by $R(\lambda, A^*)(E^*) = D(A^*)$ is $T^*(t)$-invariant. Since $T^*(t)$, being the adjoint of a positive operator, is order continuous, this implies that the band $(D(A^*))^{dd} = (E^\odot)^{dd}$ is $T^*(t)$-invariant as well.

2. The structure of $E^\odot$

In this section we will assume that $T(t)$ is a positive C_0-semigroup on a Banach lattice E.

THEOREM 2.1. *If $E^\odot$ is contained in a sublattice of E^* with order continuous norm, then $E^\odot$ is an ideal in E^*. In particular, if E^* has order continuous norm, then $E^\odot$ is a projection band.*

Proof. Let F be a sublattice of E^* with order continuous norm, containing $E^\odot$.

Step 1. First let $0 \leqslant |x^*| \leqslant y^*$ with $y^* \in E^\odot$. We will show that $x^* \in E^\odot$. Choose $\lambda_0 > 0$ such that $R(\lambda, A) \geqslant 0$ for $\lambda \geqslant \lambda_0$. Put

$$G := \{\lambda R(\lambda, A)^* y^* : \lambda \geqslant \lambda_0\}.$$

Since $y^* \in E^\odot$, this set is relatively compact subset of F, hence certainly relatively weakly compact in F. Let

$$\mathrm{sol}_F G := \{f \in F : \exists g \in G \text{ with } |f| \leqslant g\}$$

be the solid hull of G in F. Since F has order continuous norm, $\mathrm{sol}_F G$ is relatively weakly compact in F [M, Prop. 2.5.12 (iv)]. Since $E^\odot \subset F$ and $0 \leqslant |\lambda R(\lambda, A)^* x^*| \leqslant R(\lambda, A)^* |x^*| \leqslant \lambda R(\lambda, A)^* y^*$ for all $\lambda \geqslant \lambda_0$, it is clear that

$$H := \{\lambda R(\lambda, A)^* x^* : \lambda \geqslant \lambda_0\} \subset \mathrm{sol}_F G.$$

In particular, H is relatively weakly compact in F. Let z^* be any $\sigma(F, F^*)$-accumulation point of H as $\lambda \to \infty$. Then z^* is also a weak- and hence a weak*-accumulation point of H. But on the other hand, weak* $\lim_{\lambda \to \infty} \lambda R(\lambda, A)^* x^* = x^*$. Therefore necessarily $z^* = x^*$. Since $\lambda R(\lambda, A)^* x^* \in E^\odot$ for each $\lambda \geqslant \lambda_0$, it follows that x^* belongs to the weak closure of $E^\odot$. Hence $x^* \in E^\odot$.

Step 2. Suppose $|x^*| \leqslant |y^*|$ with $y^* \in E^\odot$. We will show that $x^* \in E^\odot$. By Step 1 it suffices to show that $|x^*| \in E^\odot$. Therefore we may assume that $x^* \geqslant 0$. For $\lambda \geqslant \lambda_0$ put

$$z_\lambda^* := |\lambda R(\lambda, A)^* y^*| \wedge x^*.$$

Then, since $x^* \geqslant 0$ and $\lambda R(\lambda, A)^* \geqslant 0$,

$$0 \leqslant z_\lambda^* \leqslant |\lambda R(\lambda, A)^* y^*| \leqslant \lambda R(\lambda, A)^* |y^*|,$$

and since $\lambda R(\lambda, A)^* |y^*|$ is a positive element in $E^\odot$, it follows from Step 1 that $z_\lambda^* \in E^\odot$. But since $y^* \in E^\odot$ we have $\lim_{\lambda \to \infty} |\lambda R(\lambda, A)^* y^*| = |y^*|$, and therefore

$$\lim_{\lambda \to \infty} z_\lambda^* = \lim_{\lambda \to \infty} |\lambda R(\lambda, A)^* y^*| \wedge x^* = |y^*| \wedge x^* = x^*.$$

Since $E^\odot$ is closed it follows that $x^* \in E^\odot$. This proves that $E^\odot$ is an ideal.

The second statement is a consequence of the fact that every closed ideal in a Banach lattice with order continuous norm is a projection band.

In [NP] we observed that if E is a σ-Dedekind complete Banach lattice, then the band generated by $E^\odot$ is the whole E^*. In fact, this follows from weak* $\lim_{\lambda \to \infty} \lambda R(\lambda, A)^* x^* = x^*$ and the fact that every band projection in the dual of a σ-Dedekind complete Banach lattice is weak*-sequentially continuous [AB, Thm. 13.14] (consider the band projection onto the band generated by $E^\odot$).

COROLLARY 2.2. *If E is a σ-Dedekind complete Banach lattice whose dual has order continuous norm, then $E^{\odot} = E^*$.*

An example of such a Banach lattice is $E = c_0$.

The following corollary is a converse of Theorem 2.1 in case $R(\lambda, A)$ is weakly compact for some $\lambda \in \rho(A)$ (hence for all $\lambda \in \rho(A)$). This is the case if and only if E is $\odot$-reflexive with respect to $T(t)$; see [Pa1].

COROLLARY 2.3. *If $R(\lambda, A)$ is weakly compact, then the following assertions are equivalent:*

(i) *$E^{\odot}$ is an ideal;*

(ii) *$E^{\odot}$ is contained in a sublattice with order continuous norm;*

(iii) *$E^{\odot}$ is a σ-Dedekind complete sublattice.*

Proof. (iii) $\Rightarrow$ (ii): If $E^{\odot}$ is σ-Dedekind complete then, by the weak compactness of $R(\lambda, A)$, $E^{\odot}$ actually has order continuous norm [NP]. (ii) $\Rightarrow$ (i) follows from Theorem 2.1 and (i) $\Rightarrow$ (iii) follows from the fact that the dual of a Banach lattice is always Dedekind complete.

3. Disjointness almost everywhere

Throughout this section, let $T(t)$ be a positive C_0-semigroup on a Banach lattice E. Fix a real $\lambda \in \rho(A)$ with $\lambda > \omega$, with $\omega \in \mathbb{R}$ such that $\|T(t)\| \leqslant Me^{\omega t}$ for a suitable constant $M \geqslant 1$.

We start with the simple observation that $x \in \{R(\lambda, A)x\}^{dd}$ for all $0 \leqslant x \in E$. Indeed, suppose $y \in E$ such that $y \wedge R(\lambda, A)x = 0$. Since $0 \leqslant R(\mu, A)x \leqslant R(\lambda, A)x$ for all $\mu \geqslant \lambda$, this implies that

$$y \wedge (\mu R(\mu, A)x) = 0.$$

Now it follows from $\lim_{\mu \to \infty} \mu R(\mu, A)x = x$ that $y \wedge x = 0$. This shows that

$$\{R(\lambda, A)x\}^d \subset \{x\}^d,$$

and hence $x \in \{R(\lambda, A)x\}^{dd}$.

For the adjoint semigroup the situation is different. It can happen that $x^* \perp R(\lambda, A^*)x^*$ for some $0 \leqslant x^* \in X^*$. For example, let $T(t)$ be the translation group on $E = C_0(\mathbb{R})$ and let x^* be a measure which is singular with respect to the Lebesgue measure. Then $x^* \perp L^1(\mathbb{R})$, here identifying absolutely continuous measures with their L^1-densities. But $R(\lambda, A^*)x^* \in E^{\odot} = L^1(\mathbb{R})$, so indeed $x^* \perp R(\lambda, A^*)x^*$.

As one of the results of this section we will characterize these functionals x^*

as the elements of $(E^{\odot})^d$. The following lemma is a first step towards this characterization.

We will use repeatedly the formula

$$\langle x^* \wedge y^*, x\rangle = \inf\{\langle x^*, u\rangle + \langle y^*, v\rangle : u, v \in [0, x],\ u + v = x\}, \tag{$*$}$$

valid for arbitrary $x^*, y^* \in E^*$ and $0 \leqslant x \in E$ (see e.g. [Z], Theorem 83.6).

LEMMA 3.1. *Suppose* $0 \leqslant x \in E$, $0 \leqslant x^* \in E^*$ *and* $0 \leqslant y^* \in E^*$ *satisfy* $\langle R(\lambda, A^*)x^* \wedge y^*, x\rangle = 0$. *Then, for almost all* $t \geqslant 0$ *(with respect to the Lebesgue measure) we have* $\langle T^*(t)x^* \wedge y^*, x\rangle = 0$.

Proof. The formula (*) applied to $T^*(t)x^* \wedge y^*$ shows that for $x \geqslant 0$ the function $f(t) := \langle T^*(t)x^* \wedge y^*, x\rangle$ is measurable, being the infimum of continuous functions. We must show that $f = 0$ a.e. Fix $\varepsilon > 0$. By (*), applied to $R(\lambda, A^*)x^* \wedge x^*$, it is possible to choose $u, v \in [0, x]$ such that $u + v = x$ and

$$\langle R(\lambda, A^*)x^*, u\rangle < \varepsilon,\ \langle y^*, v\rangle < \varepsilon.$$

Then

$$\int_0^\infty e^{-\lambda t}\langle T^*(t)x^* \wedge y^*, x\rangle\, dt \leqslant \int_0^\infty e^{-\lambda t}\langle T^*(t)x^*, u\rangle\, dt + \int_0^\infty e^{-\lambda t}\langle y^*, v\rangle\, dt$$
$$= \langle R(\lambda, A^*)x^*, u\rangle + \lambda^{-1}\langle y^*, v\rangle \leqslant (1 + \lambda^{-1})\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary it follows that

$$\int_0^\infty e^{-\lambda t}\langle T^*(t)x^* \wedge y^*, x\rangle\, dt = 0.$$

The lemma now follows from the fact that the integrand is a positive function.

Thus, if $R(\lambda, A^*)x^* \wedge y^* = 0$, then by the lemma for all $x \geqslant 0$ we have $\langle T^*(t)x^* \wedge y^*, x\rangle = 0$, except for t belonging to a set of measure zero. This exceptional set, however, may vary with x and therefore one cannot conclude that $T^*(t)x^* \wedge y^* = 0$ for almost all t. The following example shows that indeed this need not be the case.

EXAMPLE 3.2. Let T be the unit circle in the complex plane, which will be identified with the interval $[0, 2\pi)$, and let $C(T)$ denote the Banach lattice of continuous functions on T. Let $E = l^1([0, 2\pi); C(T))$. With the pointwise order, E is a Banach lattice. Note that $E^* = l^\infty([0, 2\pi); M(T))$, where $M(T) = C(T)^*$ is the space of bounded Borel measures on T. Define an element $x^* \in E^*$ by

$x^*(\alpha) = \delta_0 + \delta_\alpha$, where δ_α is the Dirac measure concentrated at α. Let $R(t)$ be the rotation group on $C(T)$ and define a positive C_0-group $T(t)$ on E by

$$(T(t)x)(\alpha) := R(t)(x(\alpha)).$$

Then, using the fact that the lattice operations on E are defined pointwise, for any $t \in [0, \pi)$ we have

$$\|T^*(t)x^* \wedge x^*\| \geqslant \|(T^*(t)x^* \wedge x^*)(t)\| = \|R(t)(x^*(t)) \wedge x^*(t)\|$$
$$= \|(\delta_t + \delta_{2t}) \wedge (\delta_0 + \delta_t)\| = \|\delta_t\| = 1.$$

THEOREM 3.3. *Suppose that E has a quasi-interior point, or that E^* has order continuous norm. Then* $R(\lambda, A^*)x^* \wedge y^* = 0$ $(0 \leqslant x^*, y^* \in E^*)$ *implies that* $T^*(t)x^* \wedge y^* = 0$ *for almost all* $t \geqslant 0$.

Proof. Suppose first that $u > 0$ is quasi-interior. We have by Lemma 3.1 that

$$\langle T^*(t)x^* \wedge y^*, u\rangle = 0, \quad \text{a.a } t \geqslant 0.$$

Since u is a quasi-interior point, this implies that

$$T^*(t)x^* \wedge y^* = 0, \text{ a.a } t \geqslant 0.$$

If E^* has order continuous norm, then for all $z^* \in E^*$ the closed unit ball B_E is approximately z^*-order bounded [M, Prop. 2.3.2], i.e. for all $\varepsilon > 0$ and $z^* \in E^*$ there is an $x \geqslant 0$ such that

$$B_E \subset [-x, x] + \varepsilon B_{z^*}.$$

Here B_{z^*} is the closed unit ball of the seminorm p_{z^*} defined by $p_{z^*}(x) = \langle |z^*|, |x|\rangle$. Choose $x_n \geqslant 0$ such that $B_E \subset [-x_n, x_n] + n^{-1}B_{y^*}$. By Lemma 3.1, there is a set $F_n \subset \mathbb{R}_{>0}$ of full measure such that for all $t \in F_n$ we have $\langle T^*(t)x^* \wedge y^*, x_n\rangle = 0$. Fix any $t \in F_n$. Let $y \in B_E$ arbitrary. Write $y = y_1 + y_2$ with $y_1 \in [-x_n, x_n]$, $y_2 \in n^{-1}B_{y^*}$. Then

$$\langle T^*(t)x^* \wedge y^*, |y|\rangle \leqslant \langle T^*(t)x^* \wedge y^*, |y_1|\rangle + \langle T^*(t)x^* \wedge y^*, |y_2|\rangle$$
$$\leqslant 0 + \langle y^*, |y_2|\rangle \leqslant \frac{1}{n}.$$

It follows that $\langle T^*(t)x^* \wedge y^*, |y|\rangle = 0$ for all $t \in F := \bigcap_n F_n$. Since y is arbitrary and the F_n do not depend on y, it follows that $T^*(t)x^* \wedge y^* = 0$ for $t \in F$.

COROLLARY 3.4. *Suppose $x^* \in E^*$, $y^* \in (E^{\odot})^d$ and either E^* has order continu-*

ous norm or E has a quasi-interior point. Then $T^*(t)x^* \perp y^*$ for almost all $t \geqslant 0$.

Proof. $y^* \perp E^\odot$ implies $|y^*| \perp E^\odot$, so in particular $R(\lambda, A^*)|x^*| \wedge |y^*| = 0$. Therefore $T^*(t)|x^*| \wedge |y^*| = 0$ for almost all t. But $|T^*(t)x^*| \leqslant T^*(t)|x^*|$, hence for almost all t also $|T^*(t)x^*| \wedge |y^*| = 0$.

The following theorem gives the characterization of functionals in $(E^\odot)^d$, mentioned at the beginning of this section.

THEOREM 3.5. *For $0 \leqslant x^* \in E^*$ the following statements are equivalent:*

(i) $x^* \in (E^\odot)^d$;

(ii) $R(\lambda, A^*)x^* \wedge x^* = 0$;

(iii) *For all $0 \leqslant x \in E$ we have $\langle T^*(t)x^* \wedge x^*, x\rangle = 0$ for almost all $t \geqslant 0$;*

(iv) *For all $0 \leqslant x \in E$ we have $\liminf_{t \downarrow 0} \langle T^*(t)x^* \wedge x^*, x\rangle = 0$.*

Proof. The implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv) are trivial, and (ii) $\Rightarrow$ (iii) follows from Lemma 3.1. So only (iv) $\Rightarrow$ (i) needs proof. Take $0 \leqslant x^* \in E^*$ satisfying (iv). Since $E^\odot = \overline{D(A^*)} = \overline{R(\lambda, A^*)E^*}$, it is sufficient to prove that $x^* \perp R(\lambda, A^*)y^*$ for all $y^* \in E^*$. Moreover, since

$$|R(\lambda, A^*)y^*| \leqslant R(\lambda, A^*)|y^*| \in E^\odot,$$

all we have to show is that $x^* \wedge z^\odot = 0$ for all $0 \leqslant z^\odot \in E^\odot$. To this end, fix $0 \leqslant z^\odot \in E^\odot$ and let $x_1^* \in E^*$ be any vector such that $0 \leqslant x_1^* \leqslant nx^* \wedge z^\odot$ for some number $n \in \mathbb{N}$. It follows from $0 \leqslant x_1^* \leqslant nx^*$ that

$$0 \leqslant T^*(t)x_1^* \wedge x_1^* \leqslant T^*(t)nx^* \wedge nx^* = n(T^*(t)x^* \wedge x^*),$$

so x_1^* satisfies (iv) as well. Fix $\varepsilon > 0$ and $0 \leqslant x \in E$ with $\|x\| = 1$. There exists $\delta > 0$ such that $\|T^*(t)z^\odot - z^\odot\| < \varepsilon$ for all $0 \leqslant t < \delta$. Furthermore, since $\liminf_{t \downarrow 0} \langle T^*(t)x_1^* \wedge x_1^*, x\rangle = 0$, there exists $0 < t_0 < \delta$ such that

$$0 \leqslant \langle T^*(t_0)x_1^* \wedge x_1^*, x\rangle < \varepsilon.$$

By the formula (*), there exist $0 \leqslant u, v \in E$ such that $u + v = x$ and

$$\langle T(t_0)^*x_1^*, u\rangle < \varepsilon, \ \langle x_1^*, v\rangle < \varepsilon.$$

Then

$$\langle x_1^*, u\rangle = \langle x_1^*, x\rangle - \langle x_1^*, v\rangle > \langle x_1^*, x\rangle - \varepsilon$$

and

$$\langle T^*(t_0)x_1^*, v\rangle = \langle x_1^*, x\rangle + \langle T^*(t_0)x_1^* - x_1^*, x\rangle - \langle T^*(t_0)x_1^*, u\rangle > \langle x_1^*, x\rangle - 2\varepsilon.$$

This implies that

$$\begin{aligned}\langle z^{\odot}, v\rangle &= \langle T^*(t_0)z^{\odot}, v\rangle - \langle T^*(t_0)z^{\odot} - z^{\odot}, v\rangle \\ &\geqslant \langle T^*(t_0)x_1^*, v\rangle - \|T^*(t_0)z^{\odot} - z^{\odot}\| \; \|v\| \\ &> (\langle x_1^*, x\rangle - 2\varepsilon) - \varepsilon\|v\| \geqslant \langle x_1^*, x\rangle - 3\varepsilon.\end{aligned}$$

Hence

$$\langle z^{\odot}, x\rangle = \langle z^{\odot}, u\rangle + \langle z^{\odot}, v\rangle > \langle x_1^*, u\rangle + (\langle x_1^*, x\rangle - 3\varepsilon) > \langle 2x_1^*, x\rangle - 4\varepsilon.$$

Since ε is arbitrary it follows that $\langle z^{\odot}, x\rangle \geqslant \langle 2x_1^*, x\rangle$ for all $x \geqslant 0$, i.e. $0 \leqslant 2x_1^* \leqslant z^{\odot}$. Hence, $0 \leqslant 2x_1^* \leqslant 2nx^* \wedge z^{\odot}$ and we can repeat the above argument. After doing so k times we find that $0 \leqslant 2^k x_1^* \leqslant z^{\odot}$. Hence this holds for all $k \in \mathbb{N}$, so $x_1^* = 0$. In particular, letting $x_1^* = x^* \wedge z^{\odot}$, it follows that $x^* \wedge z^{\odot} = 0$. This completes the proof.

Next we will study the behaviour of $T^*(t)$ on the disjoint complement $(E^{\odot})^d$. In general, $(E^{\odot})^d$ need not be $T^*(t)$-invariant. It may even happen that $T^*(t)E^* \subset E^{\odot}$ for all $t > 0$, e.g. if $T(t)$ is an analytic semigroup. Using the above theorem we obtain the following result.

COROLLARY 3.6. *If $T^*(t)$ is a lattice semigroup, then $(E^{\odot})^d$ is $T^*(t)$-invariant.*
Proof. If $0 \leqslant x^* \in (E^{\odot})^d$, then $R(\lambda, A^*)x^* \wedge x^* = 0$. Hence also

$$R(\lambda, A^*)T^*(t)x^* \wedge T^*(t)x^* = T^*(t)(R(\lambda, A^*)x^* \wedge x^*) = 0,$$

so $T^*(t)x^* \in (E^{\odot})^d$ by Theorem 3.5.

We note that, in particular, if $T(t)$ extends to a positive group, then $T^*(t)$ is a lattice semigroup and the above corollary applies. Furthermore we note that, as observed before, if $T^*(t)$ is a lattice semigroup, then $E^{\odot}$ is a sublattice of E^*.

The following example shows that Corollary 3.6 (and some results to follow) fail if $T^*(t)$ is not a lattice semigroup.

EXAMPLE 3.7. Let $T(t)$ be the semigroup on $E = C[0, 1]$ defined by

$$T(t)f(s) = \begin{cases} f(t+s), & t + s \leqslant 1; \\ f(1), & \text{else.} \end{cases}$$

Then one easily verifies the following facts:
(i) $E^{\odot} = L^1[0,1] \oplus \mathbb{R}\delta_1$;
(ii) $\delta_0 \perp E^{\odot}$ and $T^*(t)\delta_0 = \delta_1 \in E^{\odot}$ for all $t \geqslant 1$.

In view of Corollary 3.6 we will restrict our attention in the last part of this section mainly to the situation in which $T^*(t)$ is a lattice semigroup. We will study the occurrence of mutually disjoint elements in the orbits $\{T^*(t)x^*: t \geqslant 0\}$, where $0 \leqslant x^* \in (E^{\odot})^d$. The first result in this direction is a simple consequence of Theorem 3.3.

PROPOSITION 3.8. *Assume that E^* has order continuous norm, or that E has a quasi-interior point. Furthermore, assume that $T^*(t)$ is a lattice semigroup. Then for $0 \leqslant x^* \in (E^{\odot})^d$ we have:*

(i) *If $s \geqslant 0$ is fixed, then $T^*(t)x^* \perp T^*(s)x^*$ for almost all $t \geqslant 0$;*

(ii) *$T^*(t)x^* \perp T^*(s)x^*$ for almost all pairs $(t, s) \geqslant 0$ (with respect to the Lebesgue measure on $\mathbb{R}_+ \times \mathbb{R}_+$).*

Proof. (i) Take $s \geqslant 0$. It follows from Corollary 3.6 that $T^*(s)x^* \in (E^{\odot})^d$. Now the result follows from Theorem 3.3 (with $y^* = T^*(s)x^*$).

(ii) This follows via Fubini's theorem from (i).

Suppose that $(E^{\odot})^d \neq \{0\}$ and let $0 < x^* \in (E^{\odot})^d$ be fixed. We define

$$t_0 := \inf\{t > 0: T^*(t)x^* = 0\}.$$

If $T^*(t)x^* \neq 0$ for all $t \geqslant 0$ we put $t_0 = \infty$. If $t_0 < \infty$, it follows from the weak*-continuity of $t \mapsto T^*(t)x^*$ that $T^*(t_0)x^* = 0$; in particular $t_0 > 0$. Hence $T^*(t)x^* > 0$ for all $0 \leqslant t < t_0$ and $T^*(t)x^* = 0$ for all $t \geqslant t_0$.

We will say that a set $H \subset [0, t_0)$ *supports a disjoint system* (for x^*) if $\{T^*(t)x^*: t \in H\}$ is a disjoint system in E^*, i.e. $T^*(t)x^* \perp T^*(s)x^*$ for any two $t \neq s \in H$. In view of Proposition 3.8 one might ask whether there exist 'large' sets supporting a disjoint system. Observe already that, by Zorn's lemma, any set supporting a disjoint system is contained in a maximal one.

Let m^* denote the outer Lebesgue measure.

LEMMA 3.9. *Suppose that E^* has order continuous norm, or E has a quasi-interior point. Suppose $T^*(t)$ is a lattice semigroup and let $x^* \in (E^{\odot})^d$.*

(i) *If $H \subset [0, t_0)$ is a countable set supporting a disjoint system, and if $J \subset [0, t_0)$ is an open interval, then there exists $s \in J \backslash H$ such that $H \cup \{s\}$ supports a disjoint system.*

(ii) *If $H \subset [0, t_0)$ is a maximal set supporting a disjoint system, then H is uncountable.*

(iii) *Let $H \subset [0, t_0)$ support a disjoint system. If $T^*(t)x^* \wedge T^*(s)x^* > 0$ for some $0 < s < t$, then $m^*([0, t] \backslash H) \geqslant \frac{1}{2}s$.*

Proof. (i) For $t \in H$ let

$$F_t = \{h \geqslant 0: T^*(h)x^* \wedge T^*(t)x^* = 0\}.$$

By Proposition 3.8(i) we know that $m(\mathbb{R}_+ \backslash F_t) = 0$. Since H is countable, the set

$F = \cap\{F_t: t \in H\}$ satisfies $m(\mathbb{R}_+ \backslash F) = 0$ as well, and hence $F \cap J \neq \emptyset$. Now take any $s \in F \cap J$.

(ii) Follows immediately from (i).

(iii) Since $|T^*(t)x^*| \wedge |T^*(s)x^*| > 0$, also $|T^*(t-s+h)x^*| \wedge |T^*(h)x^*| > 0$ for all $0 \leqslant h \leqslant s$. Hence, if $h \in H \cap [0, s]$, then $h + t - s \notin H$, i.e.

$$([0, s] \cap H) + t - s \subset [0, t] \backslash H,$$

so $m^*([0, s] \cap H) \leqslant m^*([0, t] \backslash H)$. Now

$$s \leqslant m^*([0, s] \cap H) + m^*([0, s] \backslash H) \leqslant 2m^*([0, t] \backslash H),$$

so $m^*([0, t] \backslash H) \geqslant \frac{1}{2}s$.

We do not know whether a maximal set supporting a disjoint system must be measurable. This is the reason for taking the outer Lebesgue measure rather than the Lebesgue measure.

EXAMPLE 3.10. Let $T(t)$ be the rotation group on $E = C(T)$. Identifying the unit circle T with $[0, 2\pi)$, we let $x^* = \delta_0 + \delta_\pi$. Then $H = [0, \pi)$ is a maximal set supporting a disjoint system for x^*. This shows that the constant $\frac{1}{2}$ in Lemma 3.9(iii) is optimal.

THEOREM 3.11. *Suppose that E^* has order continuous norm, or E has a quasi-interior point. Suppose $T^*(t)$ is a lattice semigroup and let $x^* \in (E^{\odot})^d$.*

(i) *There exists an uncountable dense set $H \subset [0, t_0)$ supporting a disjoint system.*

(ii) *If $T(t)$ extends to a positive group, then either the orbit $\{T^*(t)x^* : t \in \mathbb{R}\}$ is a disjoint system, or $m^*(\mathbb{R} \backslash H) = \infty$ for each set $H \subset \mathbb{R}$ supporting a disjoint system.*

Proof. (i) Let $(J_n)_{n=1}^{\infty}$ be an enumeration of the open intervals in $[0, t_0)$ with rational endpoints. Using Lemma 3.9(i) we inductively construct a sequence $(t_n)_{n=1}^{\infty}$ supporting a disjoint system with $t_n \in J_n$ for all n. This sequence (t_n) is contained in some maximal H supporting a disjoint system. Clearly H is dense in $[0, t_0)$, and by Lemma 3.9(ii) H is uncountable.

(ii) Now assume in addition that $T(t)$ extends to a positive group, and that $H \subset \mathbb{R}$ supports a disjoint system with $m^*(\mathbb{R}_+ \backslash H) = K < \infty$. Then also $H_+ := H \cap \mathbb{R}_+$ supports a disjoint system and $m^*(\mathbb{R}_+ \backslash H_+) \leqslant K$. It follows from Lemma 3.9(iii) that $T^*(t)x^* \wedge T^*(s)x^* = 0$ for all $s \neq t > 2K$. Therefore, if $s \neq t$ in $\mathbb{R}$, then for n so large that $s + n > 2K$, $t + n > 2K$ we have

$$T^*(n)(T^*(t)x^* \wedge T^*(s)x^*) = T^*(t+n)x^* \wedge T^*(s+n)x^* = 0.$$

Since $T^*(n)$ is injective, this implies that $T^*(t)x^* \wedge T^*(s)x^* = 0$.

In the situation of Theorem 3.11, it is clear from (i) that $(E^{\odot})^d$ is not norm separable. So in this situation we have either $(E^{\odot})^d = \{0\}$ or $(E^{\odot})^d$ is non-separable. In this direction we can prove more, under much weaker assumptions, using a different method of proof. This is what we will do next.

First we recall some facts. Let E be a Banach lattice and $J \subset E$ an ideal. The annihilator $J^{\perp} = \{x^* \in E^*: \langle x^*, x\rangle = 0, \forall x \in J\}$ is a band in E^*, and hence we have the band decomposition $E^* = J^{\perp} \oplus (J^{\perp})^d$. Let P_J be the band projection in E^* onto $(J^{\perp})^d$.

LEMMA 3.12. *Let $J \subset E$ be an ideal and $0 \leqslant T: E \to E$ be a positive operator such that $T(J) \subset J$. Then $P_J T^* \leqslant T^* P_J$.*

Proof. Since $T(J) \subset J$ implies that $T^*(J^{\perp}) \subset J^{\perp}$, it follows that

$$T^*(I - P_J) = (I - P_J)T^*(I - P_J),$$

and so $P_J T^* P_J = P_J T^*$. Hence $P_J T^* = P_J T^* P_J \leqslant T^* P_J$.

In the following theorem, $T(t)$ is any positive C_0-semigroup on E. We do *not* assume that $T^*(t)$ be a lattice semigroup.

THEOREM 3.13. *If $(E^{\odot})^d$ contains a weak order unit, then $T^*(t)(E^*) \subset (E^{\odot})^{dd}$ for all $t > 0$.*

Proof. Let $0 \leqslant w^* \in (E^{\odot})^d$ be a weak order unit. Fix $0 \leqslant x^* \in E^*$ and $0 \leqslant x \in E$. Let J be the closed ideal in E generated by the orbit $\{T(t)x: t \geqslant 0\}$. Then J is $T(t)$-invariant and has a quasi-interior point $0 \leqslant u \in J$. By Lemma 3.1, $0 \leqslant w^* \in (E^{\odot})^d$ implies that $\langle T^*(t)x^* \wedge w^*, u\rangle = 0$ for almost all $t \geqslant 0$. Since

$$0 \leqslant P_J(T^*(t)x^*) \wedge w^* \leqslant T^*(t)x^* \wedge w^*,$$

it follows that $\langle P_J(T^*(t)x^*) \wedge w^*, u\rangle = 0$ a.e., and hence $P_J(T^*(t)x^*) \wedge w^* \in J^{\perp}$ a.e. But also $P_J(T^*(t)x^*) \wedge w^* \in (J^{\perp})^d$, so $P_J(T^*(t)x^*) \wedge w^* = 0$ a.e., hence $P_J(T^*(t)x^*) \in (E^{\odot})^{dd}$ a.e. Now observe that, if $t \geqslant 0$ is such that $P_J(T^*(t)x^*) \in (E^{\odot})^{dd}$, then by Lemma 3.12,

$$P_J(T^*(t + s)x^*) = P_J(T^*(s)T^*(t)x^*) \leqslant T^*(s)P_J(T^*(t)x^*).$$

Also, as observed in Section 1, $(E^{\odot})^{dd}$ is $T^*(t)$-invariant. Combining these facts, we conclude that $P_J(T^*(t)x^*) \in (E^{\odot})^{dd}$ for *all* $t > 0$. Therefore, $P_J(T^*(t)x^* \wedge w^*) = 0$, i.e., $T^*(t)x^* \wedge w^* \in J^{\perp}$ for all $t > 0$, which implies in particular that $\langle T^*(t)x^* \wedge w^*, x\rangle = 0$ for all $t > 0$. Since $0 \leqslant x \in E$ was arbitrary, it follows that $T^*(t)x^* \wedge w^* = 0$ for all $t > 0$, i.e., $T^*(t)x^* \in (E^{\odot})^{dd}$ for all $t > 0$.

Together with Theorem 2.1 this implies:

COROLLARY 3.14. *Suppose E^* has order continuous norm. If $(E^\odot)^d$ contains a weak order unit, then $T^*(t)(E^*) \subset E^\odot$ for all $t > 0$, i.e. $T^*(t)$ is strongly continuous for $t > 0$.*

COROLLARY 3.15. *Suppose E^* has order continuous norm and suppose $T(t)$ extends to a (not necessarily positive) group. Then either $E^* = E^\odot$ or $(E^\odot)^d$ does not contain a weak order unit.*

COROLLARY 3.16. *Suppose $T^*(t)$ is a lattice semigroup. Then either $(E^\odot)^d = \{0\}$ or $(E^\odot)^d$ does not contain a weak order unit.*

Proof. Suppose $(E^\odot)^d$ contains a weak order unit. By Theorem 3.13, $T^*(t)(E^*) \subset (E^\odot)^{dd}$ for all $t > 0$. It follows from Corollary 3.6 that $(E^\odot)^d$ is $T^*(t)$-invariant, and hence $T^*(t)((E^\odot)^d) = \{0\}$ for all $t > 0$. From the weak*-continuity of $t \mapsto T^*(t)x^*$ it now follows that $(E^\odot)^d = \{0\}$.

The preceding results can be regarded as lattice versions of the following result proved in [Ne]: If $T(t)$ is a C_0-semigroup on a Banach space X such that $X^*/X^\odot$ is separable, then $T(t)(X^*) \subset X^\odot$ for all $t > 0$, i.e. $T^*(t)$ is strongly continuous for $t > 0$. In particular, if $T(t)$ extends to a group, then either $X^\odot = X^*$ or $X^*/X^\odot$ is non-separable.

In the setting of Corollary 3.15, one might wonder when exactly one has $E^\odot = E^*$. In this direction, we can prove:

PROPOSITION 3.17. *Let $E = C_0(\Omega)$ with Ω locally compact Hausdorff, and let $T(t)$ be a positive C_0-group on E. If $E^\odot = E^*$ then $T(t)$ is a multiplication group.*

Proof. Since each operator $T^*(t)$ is a lattice isomorphism, atoms in $M(\Omega) = (C_0(\Omega))^*$ are mapped to atoms. Hence, for each $\omega \in \Omega$ we have $T^*(t)\delta_\omega = \phi_\omega(t)\delta_{\omega(t)}$, say. Here δ_ω is the Dirac measure at ω. By the strong continuity of $t \mapsto T^*(t)\delta_\omega$, we must have that $\omega(t) = \omega$, so $T^*(t)\delta_\omega = \phi_\omega(t)\delta_\omega$. For $f \in C_0(\Omega)$ one then has

$$(T(t)f)(\omega) = \langle \delta_\omega, T(t)f \rangle = \phi_\omega(t)\langle f, \delta_\omega \rangle = \phi_\omega(t)f(\omega).$$

Every multiplication group on a real Banach lattice E has a bounded generator [Na, Proposition. C-II.5.16]. If E is complex, then a positive semigroup leaves invariant the real part of E. Therefore, both in the real and complex case, from the above results we conclude:

COROLLARY 3.18. *Let $T(t)$ be a positive C_0-group with unbounded generator on the Banach lattice $E = C_0(\Omega)$. Then $(E^\odot)^d$ does not contain a weak order unit.*

4. Limes superior estimates

We start in this section with an arbitrary C_0-semigroup $T(t)$ on a Banach space X. We choose $M \geqslant 1$ and $\omega \in \mathbb{R}$ such that $\|T(t)\| \leqslant Me^{\omega t}$. It is our objective to study the quantity $\|T^*(t)x^* - x^*\|$ as $t \downarrow 0$ for $x^* \in X^*$. Our first results are general limes superior estimates, which we will improve later in the context of positive semigroups.

For $x^* \in X^*$ define

$$\rho(x^*) := \limsup_{t \downarrow 0} \|T^*(t)x^* - x^*\|.$$

It is clear that ρ defines a seminorm on X^*. Note that $\rho(x^* + x^{\odot}) = \rho(x^*)$ for all $x^{\odot} \in X^{\odot}$ and $x^* \in X^*$. In particular, $\rho(x^*) = 0$ if and only if $x^* \in X^{\odot}$. Furthermore,

$$\rho(x^*) \leqslant \limsup_{t \downarrow 0} (\|T^*(t)\| + 1)\|x^*\| \leqslant (M + 1)\|x^*\|$$

for all $x^* \in X^*$.

Since $X^{\odot}$ is a closed subspace of X^*, the quotient space $X^*/X^{\odot}$ is a Banach space. Let $q: X^* \to X^*/X^{\odot}$ be the quotient map. The following result shows that the seminorm ρ is actually equivalent to the quotient norm on $X^*/X^{\odot}$.

THEOREM 4.1. *For all* $x^* \in X^*$ *we have* $\|qx^*\| \leqslant \rho(x^*) \leqslant (M + 1)\|qx^*\|$.

Proof. For arbitrary $x^* \in X^*$ and $x^{\odot} \in X^{\odot}$ we have

$$\rho(x^*) = \rho(x^* + x^{\odot}) \leqslant (M + 1)\|x^* + x^{\odot}\|.$$

By taking the infimum over all $x^{\odot} \in X^{\odot}$ we obtain $\rho(x^*) \leqslant (M + 1)\|qx^*\|$.

For the converse, we recall that for any $\tau > 0$ we have weak* $\int_0^\tau T^*(t)x^* \, dt \in X^{\odot}$. Therefore,

$$\begin{aligned} \|qx^*\| &\leqslant \left\| \frac{1}{\tau} \text{ weak*} \int_0^\tau T^*(t)x^* \, dt - x^* \right\| = \frac{1}{\tau} \left\| \text{weak*} \int_0^\tau (T^*(t)x^* - x^*) dt \right\| \\ &\leqslant \frac{1}{\tau} \int_0^\tau \|T^*(t)x^* - x^*\| \, dt \leqslant \sup_{0 \leqslant t \leqslant \tau} \|T^*(t)x^* - x^*\|. \end{aligned}$$

Hence,

$$\|qx^*\| \leqslant \inf_{\tau > 0} \left(\sup_{0 \leqslant t \leqslant \tau} \|T^*(t)x^* - x^*\| \right) = \rho(x^*).$$

We mention an immediate consequence of the above theorem.

COROLLARY 4.2. *Let* $X^{\odot} \subset Y$, *with* Y *a complemented subspace of* X^*, *say* $X^* = Y \oplus Z$. *Then there is a constant* $C > 0$ *such that for all* $x^* \in Z$ *we have*

$$\limsup_{t \downarrow 0} \|T^*(t)x^* - x^*\| \geqslant C\|x^*\|.$$

Proof. Since $X^{\odot} \subset Y$, the formula $|||x^*||| := \|qx^*\|$ defines a norm on Z which satisfies $|||x^*||| = \inf_{x^{\odot} \in X^{\odot}} \|x^* - x^{\odot}\| \geqslant \inf_{y \in Y} \|x^* - y\|$. But $X^*/Y \simeq Z$ and consequently $|||x^*||| \geqslant C\|x^*\|$. Now we can apply Theorem 4.1.

On the quotient space $X^*/X^{\odot}$ we can define a quotient semigroup $T_q^*(t)$ via the formula

$$T_q^*(t)qx^* := q(T^*(t)x^*).$$

Using the equivalence in Theorem 4.1, we can investigate some properties of this quotient semigroup via the seminorm ρ. For this purpose, the following result turns out to be useful.

LEMMA 4.3. *Let* $[a, b] \subset \mathbb{R}$ *be a closed interval and* $f: [a, b] \to X^*$ a weak*-*continuous function. Then* $t \mapsto \rho(f(t))$ *is a bounded Borel function on* $[a, b]$ *and*

$$\rho\left(\text{weak* } \int_a^b f(t)\,\mathrm{d}t\right) \leqslant \int_a^b \rho(f(t))\,\mathrm{d}t.$$

Proof. For $n \in \mathbb{N}$, $n > 0$, define

$$\rho_n(x^*) := \sup_{0 \leqslant t \leqslant 1/n} \|T^*(t)x^* - x^*\|, \; x^* \in X^*.$$

Each ρ_n is a seminorm on X^* and $\rho_n(x^*) \downarrow \rho(x^*)$ for all $x^* \in X^*$. Note that

$$\begin{aligned} \rho_n(x^*) &= \sup_{0 \leqslant t \leqslant 1/n} \left(\sup_{\|x\| \leqslant 1} |\langle T^*(t)x^* - x^*, x\rangle| \right) \\ &= \sup_{0 \leqslant t \leqslant 1/n} \left(\sup_{\|x\| \leqslant 1} |\langle x^*, (T(t) - I)x\rangle| \right) \\ &= \sup\{|\langle x^*, y\rangle| : y \in D_n\}, \end{aligned}$$

where $D_n = \bigcup_{0 \leqslant t \leqslant 1/n} (T(t) - I)B_X$, B_X being the closed unit ball of X. Hence, $\rho_n(f(t)) = \sup_{x \in D_n} |\langle f(t), x\rangle|$ for all $a \leqslant t \leqslant b$. Being the pointwise supremum of continuous functions, $\rho_n(f(\cdot))$ is lower semi-continuous. Since $\rho_n(f(t)) \downarrow \rho(f(t))$ for all $a \leqslant t \leqslant b$, it follows that $\rho(f(\cdot))$ is a Borel function.

For $x \in D_n$ we have

$$\left|\left\langle \text{weak}^* \int_a^b f(t)\,dt, x\right\rangle\right| = \left|\int_a^b \langle f(t), x\rangle\,dt\right|$$

$$\leqslant \int_a^b |\langle f(t), x\rangle|\,dt \leqslant \int_a^b \rho_n(f(t))\,dt,$$

and so

$$\rho\left(\text{weak}^* \int_a^b f(t)\,dt\right) \leqslant \rho_n\left(\text{weak}^* \int_a^b f(t)\,dt\right)$$

$$= \sup_{x \in D_n} \left|\left\langle \text{weak}^* \int_a^b f(t)\,dt, x\right\rangle\right| \leqslant \int_a^b \rho_n(f(t))\,dt.$$

Finally, it follows from the monotone convergence theorem that

$$\int_a^b \rho_n(f(t))\,dt \downarrow \int_a^b \rho(f(t))\,dt.$$

This concludes the proof.

The above lemma can be used to prove the following property of the seminorm ρ.

PROPOSITION 4.4. *For all $x^* \in X^*$ we have*

$$\rho(x^*) \leqslant \limsup_{t \downarrow 0} \rho(T^*(t)x^* - x^*).$$

In particular, if $x^* \in X^*$ is such that $\lim_{t \downarrow 0} \rho(T^*(t)x^* - x^*) = 0$, then $\rho(x^*) = 0$, i.e., $x^* \in X^{\odot}$.

Proof. For all $x^* \in X^*$ and $\tau > 0$ we have, using Lemma 4.3,

$$\rho(x^*) = \rho\left(\frac{1}{\tau}\text{weak}^* \int_0^\tau (T^*(t)x^* - x^*)\,dt\right)$$

$$\leqslant \frac{1}{\tau}\int_0^\tau \rho(T^*(t)x^* - x^*)\,dt \leqslant \sup_{0 < t \leqslant \tau} \rho(T^*(t)x^* - x^*).$$

A combination of this result with Theorem 4.1 yields the following:

COROLLARY 4.5. *If $\lim_{t \downarrow 0} \|T_q^*(t)qx^* - qx^*\| = 0$, then $qx^* = 0$.*

Thus the only element in $X^*/X^\odot$ whose $T_q^*(t)$-orbit is strongly continuous, is the zero element. This result was first proved in [Ne]. The (more complicated) proof given there shows that in fact the following stronger result is true: if the $T_q^*(t)$-orbit of some qx^* is norm-separable in $X^*/X^\odot$, then it is identically zero for $t > 0$.

We now return to the Banach lattice case. In Theorem 0.1, $E^\odot$ is complemented in E^* and therefore we can already conclude from Theorem 4.1 that the limes superior estimate must hold with some constant C. In general $E^\odot$ is not complemented, but we always have a direct sum decomposition of E^* into the band generated by $E^\odot$ and the disjoint complement of $E^\odot$ (which of course may be $\{0\}$). Therefore Corollary 4.2 can be applied and we get a constant $C > 0$ such that for all $x^* \perp E^\odot$ we have

$$\limsup_{t\downarrow 0} \|T^*(t)x^* - x^*\| \geqslant C\|x^*\|.$$

The following theorem shows that in fact we can achieve $C = 2$.

THEOREM 4.6. *If* $x^* \in (E^\odot)^d$, *then* $\limsup_{t\downarrow 0} \|T^*(t)x^* - x^*\| \geqslant 2\|x^*\|$.

Proof. First we observe that for $x^* \in E^*$ and $0 \leqslant x \in E$,

$$\liminf_{t\downarrow 0} \langle |T^*(t)x^*|, x\rangle \geqslant \langle |x^*|, x\rangle.$$

Indeed, if $|y| \leqslant x$, then

$$\liminf_{t\downarrow 0} \langle T^*(t)|x^*|, x\rangle \geqslant \liminf_{t\downarrow 0} |\langle T^*(t)x^*, y\rangle| = \lim_{t\downarrow 0} |\langle T^*(t)x^*, y\rangle| = |\langle x^*, y\rangle|,$$

and hence

$$\liminf_{t\downarrow 0}\langle |T^*(t)x^*|, x\rangle| \geqslant \sup\{|\langle x^*, y\rangle| : |y| \leqslant x\} = \langle |x^*|, x\rangle.$$

Now take $x^* \in (E^\odot)^d$ and $0 \leqslant x \in E$ with $\|x\| = 1$. From Lemma 3.1 we know that $\langle T^*(t)|x^*| \wedge |x^*|, x\rangle = 0$ for almost all $t \geqslant 0$, and hence $\langle |T^*(t)x^*| \wedge |x^*|, x\rangle = 0$ a.e. Using the lattice identity [AB, Theorem 1.4(4)]

$$2(|T^*(t)x^*| \wedge |x^*|) = |T^*(t)x^*| + |x^*| - ||T^*(t)x^*| - |x^*||,$$

we see that, for almost every $t \geqslant 0$,

$$\begin{aligned}\|T^*(t)x^* - x^*\| &\geqslant \langle |T^*(t)x^* - x^*|, x\rangle \geqslant \langle ||T^*(t)x^*| - |x^*||, x\rangle \\ &= \langle |T^*(t)x^*|, x\rangle + \langle |x^*|, x\rangle.\end{aligned}$$

This implies that

$$\limsup_{t\downarrow 0} \|T^*(t)x^* - x^*\| \geqslant \liminf_{t\downarrow 0} \langle |T^*(t)x^*|, x\rangle + \langle |x^*|, x\rangle \geqslant 2\langle |x^*|, x\rangle.$$

Since $0 \leqslant x \in E$ of norm one is arbitrary, the result follows.

If E^* has order continuous norm, then by Theorem 2.1 $E^{\odot}$ is a projection band. Let π be the band projection onto its disjoint complement.

COROLLARY 4.7. *If E^* has order continuous norm, then*

$$2\|\pi x^*\| \leqslant \limsup_{t\downarrow 0} \|T^*(t)x^* - x^*\| \leqslant (M+1)\|\pi x^*\|.$$

In particular, if $M = 1$, i.e., if $\lim_{t\downarrow 0}\|T(t)\| = 1$, then $\limsup_{t\downarrow 0}\|T^*(t)x^* - x^*\| = 2\|\pi x^*\|$.

If x^* is contained in the band generated by $E^{\odot}$ but not contained in $E^{\odot}$ itself, then the limes superior can be anything between 0 and 2, as is shown by the following example.

EXAMPLE 4.8. Let $E = L^1(\mathbb{R})$, $T(t)$ the translation group on E. Let $f \in C_0(\mathbb{R})$ be of norm one such that $f = 0$ on $[-1, 1]$. Let $0 \leqslant \alpha \leqslant 1$ and define $g \in E^* = L^{\infty}(\mathbb{R})$ by

$$g(s) := \begin{cases} f(s), & |s| > 1; \\ \alpha, & s \in [0,\ 1]; \\ -\alpha, & s \in [-1,\ 0]. \end{cases}$$

Then $\|g\| = 1$, g belongs to the band generated by $E^{\odot}$, and $\limsup_{t\downarrow 0}\|T^*(t)g - g\| = 2\alpha$.

5. References

[AB] C. D. Aliprantis and O. Burkinshaw, *Positive Operators*, Pure and Applied Math. 119, Academic Press, 1985.

[BB] P. L. Butzer, and H. Berens, *Semigroups of Operators and Approximation*, Springer-Verlag, New York (1967).

[Cl] Ph. Clément, O. Diekmann, M. Gyllenberg, H. J. A. M. Heijmans and H. R. Thieme, Perturbation theory for dual semigroups, Part I: The sun-reflexive case, *Math. Ann.* 277 (1987) 709–725; Part II: Time-dependent perturbations in the sun-reflexive case, *Proc. Roy. Soc. Edinb.* 109A (1988) 145–172; Part III: Nonlinear Lipschitz perturbations in the sun-reflexive case, In: G. Da Prato, M. Iannelli (eds.), *Volterra Integro Differential Equations in Banach Spaces and Applications*, Longman, (1989) 67–89; Part IV: The intertwining formula and the canonical pairing, In: *Semigroup Theory and Applications*, Lecture Notes in Pure and Applied Mathematics, Vol. 116, Marcel Dekker Inc., New York-Basel (1989); Part V: Variation of constants formulas, In: *Semigroup Theory and Evolution Equations*, Lecture Notes in Pure and Applied Mathematics, Vol. 135, Marcel Dekker Inc., New York-Basel (1991).

[GN] A. Grabosch and R. Nagel, Order structure of the semigroup dual: a counterexample, *Indag. Math.* 92 (1989) 199–201.

[GM] C. C. Graham and O. C. McGehee, *Essays in Commutative Harmonic Analysis*, Springer-Verlag, New York, Heidelberg, Berlin (1979).

[M] P. Meyer-Nieberg, *Banach Lattices*, Springer-Verlag, Berlin, Heidelberg, New York (1991).

[MG] H. Milicer-Gruzewska, Sur la continuité de la variation, *C. R. Soc. Sci. de Varsovie*, 21 (1928) 164–177.

[Na] R. Nagel (ed.), *One-parameter Semigroups of Positive Operators*, Springer Lect. Notes in Math. 1184 (1986).

[Ne] J. M. A. M. van Neerven, *The Adjoint of a Semigroup of Linear Operators*, Lect. Notes in Math. 1529, Springer-Verlag, Berlin, Heidelberg, New York (1992).

[NP] J. M. A. M. van Neerven and B. de Pagter, Certain semigroups on Banach function spaces and their adjoints, In: *Semigroup Theory and Evolution Equations*, Lecture Notes in Pure and Applied Mathematics, Vol. 135, Marcel Dekker Inc., New York-Basel (1991).

[Pa1] B. de Pagter, A characterization of sun-reflexivity, *Math. Ann.* 283 (1989) 511–518.

[Pa2] B. de Pagter, A Wiener-Young-type theorem for dual semigroups, *Appl. Math.* 27 (1992) 101–109.

[Ph] R. S. Phillips, The adjoint semi-group, *Pac. J. Math.* 5 (1955) 269–283.

[Pl] A. Plessner, Eine Kennzeichnung der totalstetigen Funktionen, *J. f. Reine u. Angew. Math.* 60 (1929) 26–32.

[S] H. H. Schaefer, *Banach Lattices and Positive Operators*, Springer Verlag, Berlin, Heidelberg, New York (1974).

[WY] N. Wiener and R. C. Young, The total variation of $g(x+h) - g(x)$, *Trans. Am. Math. Soc.* 35 (1933) 327–340.

[Z] A. C. Zaanen, *Riesz Spaces II*, North Holland, Amsterdam (1983).